

Euler characteristics for the Hodge splitting in the rational homology and homotopy of high dimensional string links

Paul Arnaud Songhafouo Tsopméné
Victor Turchin

Abstract

The current paper is the second one of our project, which is an investigation of spaces of high dimensional string links. In the first one we showed that when the dimensions are in the stable range, the rational homology and homotopy of these latter spaces can be calculated as the homology of a direct sum of certain finite colored graph-complexes that we described explicitly. In this paper we compute the generating function of the Euler characteristics of the summands in the homological and homotopical splitting. As a byproduct result of these computations we also determine the supercharacter of the symmetric group action on the positive arity components of the modular envelop of L_∞ .

0 Introduction

Let $\text{Emb}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ be the space of smooth embeddings $f: \coprod_{i=1}^r \mathbb{R}^{m_i} \hookrightarrow \mathbb{R}^d$ that coincide outside a compact set with a fixed linear embedding $\iota: \coprod_{i=1}^r \mathbb{R}^{m_i} \hookrightarrow \mathbb{R}^d$ affine on each component. Such embeddings are called *high dimensional string links*. Let also $\text{Imm}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ be the space of smooth immersions $\coprod_{i=1}^r \mathbb{R}^{m_i} \looparrowright \mathbb{R}^d$ with the same behavior at infinity. One has an obvious inclusion $\text{Emb}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d) \hookrightarrow \text{Imm}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$. The homotopy fiber of this inclusion over ι , denoted $\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$, is the space that we studied in [22] and we continue to study in this paper.

In [22], we proved that when the dimensions are in the stable range $d > 2\max\{m_i \mid 1 \leq i \leq r\} + 1$, the rational homology and homotopy of $\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ can be described as the homology of certain explicit complexes. The complexes themselves split into a direct sum of finite complexes. Here we prove Theorem 0.1 and Theorem 0.3, which compute the generating function of the Euler characteristics of the summands respectively in the homological and homotopical splitting. In fact in [22] we defined two types of complexes computing the rational homology and homotopy groups of $\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$. The first type, that we call *Koszul complexes*, are built up from the (co)homology of configuration spaces of points in \mathbb{R}^{m_i} and in \mathbb{R}^d . These complexes we use for our main computations. The second type of complexes, that we call *hairy graph-complexes*, are essentially built up from the positive arity components of the modular envelop¹ $\mathbf{Mod}(L_\infty)$ of the L_∞ operad. Thus, as a byproduct result of our computations we also determine in Theorem 0.4 the *supercharacter* of the symmetric group action on the positive arity components of $\mathbf{Mod}(L_\infty)$. Notice that if one tries to obtain this result directly from the hairy graph-complexes (equivalently looking at the components of $\mathbf{Mod}(L_\infty)$), the direct computations of the supercharacter would be much harder. Such computations were done in case of arity zero by T. Willwacher and M. Živković in [25]. In that arity $\mathbf{Mod}(L_\infty)$ is the well known Kontsevich graph-complex associated to the commutative operad [16]. Notice also that our approach works only for the positive arity components of $\mathbf{Mod}(L_\infty)$.

¹For the notion of a modular operad and modular envelop, sometimes called modular closure, see [10, 19, 13].

Before stating our results, we have to recall Theorem 0.1 and Theorem 0.2 of [22], which are the starting point in this paper. We need to first define two categories: \mathbf{Rmod}_Ω and \mathbf{Rmod}_Γ . Let Ω denote the category of finite unpointed sets and surjections. Define a *right Ω -module* as a contravariant functor from Ω to any given category. The category of right Ω -modules in chain complexes over rationals is denoted \mathbf{Rmod}_Ω . This category can be endowed with several model structures. We choose the one called *projective model structure*. In that model, weak equivalences are quasi-isomorphisms, fibrations are degreewise surjective maps [14]. Given two objects P and Q in \mathbf{Rmod}_Ω , we write $\mathbf{Rmod}_\Omega(P, Q)$ for the space (chain complex) of morphisms between them, and we write $h\mathbf{Rmod}_\Omega(P, Q)$ for the derived mapping space. For specific computations we will need to apply this construction only to Ω -modules with zero differential, in which case $h\mathbf{Rmod}_\Omega(-, -)$ can be expressed as a product of certain Ext groups.

For a pointed topological space X , define the functor $X^{\wedge \bullet}: \Omega \rightarrow \mathbf{Top}$ from Ω to topological spaces by

$$X^{\wedge \bullet}(\{1, \dots, k\}) = X^{\wedge k} = \underbrace{X \wedge \dots \wedge X}_k \quad (\text{here “}\wedge\text{” is the smash product operation}).$$

By $X^{\wedge 0}$ we mean the two-point space – zero dimensional pointed sphere S^0 . For a morphism f in Ω , $X^{\wedge \bullet}(f)$ is induced by the diagonal maps. One can easily check that $X^{\wedge \bullet}$ is a right Ω -module. Thus, if $\tilde{C}_*(-)$ denotes the reduced chain complex functor, the functors $\tilde{C}_*(X^{\wedge \bullet})$ and $\tilde{H}_*(X^{\wedge \bullet})$ (with zero differential) are objects of \mathbf{Rmod}_Ω . Note that in case X is a suspension, any strict surjection acts on $\tilde{H}_*(X^{\wedge \bullet})$ as a zero map.

Let Γ be the category of pointed sets and pointed maps. The category of contravariant functors from Γ to chain complexes is denoted \mathbf{Rmod}_Γ . Objects of that category are called *right Γ -modules*. As an example of a right Γ -module, we have the homology $H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$, $d \geq 2$, where $C(k, \mathbb{R}^d)$ denotes the configuration space of k labeled points in \mathbb{R}^d . One can see that $H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ is indeed a right Γ -module. Firstly, since there is a morphism $\text{Com} \rightarrow H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ of operads from the commutative operad $\text{Com} = H_0(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ to the homology $H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q}) = H_*(B_d(\bullet), \mathbb{Q})$ of the little d -disks operad, it follows that $H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ is an *infinitesimal bimodule* (see [1, Definition 3.8] or [23, Definition 4.1]) over Com . Secondly it is well known that the category of infinitesimal bimodules over Com is equivalent to the category of right Γ -modules ([1, Corollary 4.10] or [23, Lemma 4.3]). One can also show that the sequence $\mathbb{Q} \otimes \pi_* C(\bullet, \mathbb{R}^d)$, $d \geq 3$, has a natural structure of a right Γ -module.

The two categories \mathbf{Rmod}_Ω and \mathbf{Rmod}_Γ we just defined are equivalent. To prove it, Pirashvili [21] constructed a functor $\text{cr}: \mathbf{Rmod}_\Omega \rightarrow \mathbf{Rmod}_\Gamma$, called *cross effect*, and showed that it is actually an equivalence of categories. Let

$$\hat{H}_*(C(\bullet, \mathbb{R}^d), \mathbb{Q}) \text{ (respectively } \mathbb{Q} \otimes \hat{\pi}_* C(\bullet, \mathbb{R}^d) \text{)} \quad (0.1)$$

denote the cross effect of $H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ (respectively the cross effect of $\mathbb{Q} \otimes \pi_* C(\bullet, \mathbb{R}^d)$).

Theorems 0.1 and 0.2 in [22] express the rational homology and homotopy groups of $\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ as the homology groups of derived mapping complexes of right Ω -modules. More precisely, they say that for $d > 2\max\{m_i \mid 1 \leq i \leq r\} + 1$, there are isomorphisms:

$$H_*(\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d), \mathbb{Q}) \cong H\left(h\mathbf{Rmod}_\Omega\left(\tilde{H}_*((\vee_{i=1}^r S^{m_i})^{\wedge \bullet}, \mathbb{Q}), \hat{H}_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})\right)\right). \quad (0.2)$$

$$\mathbb{Q} \otimes \pi_* \overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d) \cong H\left(h\mathbf{Rmod}_\Omega\left(\tilde{H}_*((\vee_{i=1}^r S^{m_i})^{\wedge \bullet}, \mathbb{Q}), \mathbb{Q} \otimes \hat{\pi}_* C(\bullet, \mathbb{R}^d)\right)\right). \quad (0.3)$$

Since $\vee_{i=1}^r S^{m_i}$ is a suspension, any strict surjection acts on $\tilde{H}_*((\vee_{i=1}^r S^{m_i})^{\wedge \bullet}, \mathbb{Q})$ as zero. This implies that

the latter Ω -module splits as follows:

$$\tilde{H}_*((\bigvee_{i=1}^r S^{m_i})^{\wedge \bullet}, \mathbb{Q}) \cong \bigoplus_{s_1, \dots, s_r \geq 0} Q_{s_1 \dots s_r}^{m_1 \dots m_r}, \quad (0.4)$$

where $Q_{s_1 \dots s_r}^{m_1 \dots m_r}$ is the right Ω -module defined by

$$Q_{s_1 \dots s_r}^{m_1 \dots m_r}(k) = \begin{cases} 0 & \text{if } k \neq s_1 + \dots + s_r; \\ \text{Ind}_{\Sigma_{s_1} \times \dots \times \Sigma_{s_r}}^{\Sigma_k} \tilde{H}_*(S^{s_1 m_1 + \dots + s_r m_r}; \mathbb{Q}) & \text{if } k = s_1 + \dots + s_r. \end{cases} \quad (0.5)$$

Consider now $\hat{H}_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ and $\mathbb{Q} \otimes \hat{\pi}_* C(\bullet, \mathbb{R}^d)$ that appear in (0.2) and (0.3) respectively. One has the splittings

$$\hat{H}_*(C(\bullet, \mathbb{R}^d), \mathbb{Q}) = \prod_{t \geq 0} \hat{H}_{t(d-1)}(C(\bullet, \mathbb{R}^d), \mathbb{Q}), \quad (0.6)$$

and

$$\mathbb{Q} \otimes \hat{\pi}_* C(\bullet, \mathbb{R}^d) = \prod_{t \geq 0} \mathbb{Q} \otimes \hat{\pi}_{t(d-2)+1} C(\bullet, \mathbb{R}^d). \quad (0.7)$$

Combining (0.2), (0.3), (0.4), (0.6), and (0.7), we get the following splittings

$$\begin{aligned} H_*(\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d), \mathbb{Q}) &\cong \prod_{s_1, \dots, s_r, t} \text{hRmod}_{\Omega} \left(Q_{s_1 \dots s_r}^{m_1 \dots m_r}, \hat{H}_{t(d-1)}(C(\bullet, \mathbb{R}^d), \mathbb{Q}) \right) \\ &\cong \bigoplus_{s_1, \dots, s_r, t} \text{hRmod}_{\Omega} \left(Q_{s_1 \dots s_r}^{m_1 \dots m_r}, \hat{H}_{t(d-1)}(C(\bullet, \mathbb{R}^d), \mathbb{Q}) \right). \end{aligned} \quad (0.8)$$

$$\begin{aligned} \mathbb{Q} \otimes \pi_*(\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)) &\cong \prod_{s_1, \dots, s_r, t} \text{hRmod}_{\Omega} \left(Q_{s_1 \dots s_r}^{m_1 \dots m_r}, \mathbb{Q} \otimes \hat{\pi}_{t(d-2)+1} C(\bullet, \mathbb{R}^d) \right) \\ &\cong \bigoplus_{s_1, \dots, s_r, t} \text{hRmod}_{\Omega} \left(Q_{s_1 \dots s_r}^{m_1 \dots m_r}, \mathbb{Q} \otimes \hat{\pi}_{t(d-2)+1} C(\bullet, \mathbb{R}^d) \right). \end{aligned} \quad (0.9)$$

The product is replaced by the direct sum because only finitely many factors contribute for any given degree. We use here that $d > 2\max\{m_i \mid 1 \leq i \leq r\} + 1$. This can be seen from graph-complexes we explicitly described in [22, Section 2], see [22, Remark 2.4]. (For (0.9) this is true even for a weaker constraint $d > \max\{m_i \mid 1 \leq i \leq r\} + 2$. Moreover, we conjecture in [22, Section 3] that (0.9) holds always in that range for $* \geq 0$.)

We now state the first result of this paper. For $s_1, \dots, s_r, t \geq 0$, let $\mathcal{X}_{s_1 \dots s_r t}$ be the Euler characteristic of the summand of (0.8) indexed by s_1, \dots, s_r, t . The associated generating function is

$$F_{m_1, \dots, m_r, d}^H(x_1, \dots, x_r, u) = \sum_{s_1, \dots, s_r, t \geq 0} \mathcal{X}_{s_1 \dots s_r t} x_1^{s_1} \dots x_r^{s_r} u^t. \quad (0.10)$$

Let $\Gamma(-)$ denote the gamma function, and let $\mu(-)$ denote the standard Möbius function. Given a variable x and an integer $l \geq 1$, let $E_l(x)$ denote the sum

$$E_l(x) = \frac{1}{l} \sum_{p|l} \mu(p) x^{\frac{l}{p}}. \quad (0.11)$$

The following result computes the generating function (0.10).

Theorem 0.1. Assume that $d > 2\max\{m_i \mid 1 \leq i \leq r\} + 1$. The generating function (0.10) is given by the formula

$$F_{m_1, \dots, m_r, d}^H(x_1, \dots, x_r, u) = \prod_{l=1}^{+\infty} \frac{\Gamma((-1)^{d-1} E_l(\frac{1}{u}) - \sum_{i=1}^r (-1)^{m_i-1} E_l(x_i))}{((-1)^{d-1} l u^l)^{\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i)} \Gamma((-1)^{d-1} E_l(\frac{1}{u}))}, \quad (0.12)$$

where each factor is understood as the asymptotic expansion of the underlying function when u is complex and $(-1)^{d-1} u^l \rightarrow +0$ and x_1, \dots, x_r are considered as fixed parameters.

Remark 0.2. When $r = 1$, the formula (0.12) coincides with that of [2, Theorem 6.1].

Using formula (0.12) we also compute the generating function of the Hodge splitting in the rational homotopy, and this gives our second result. Before we state it, we need a couple of definitions. Let B_p denote the p th Bernoulli number, so that

$$\sum_{p \geq 1} B_p x^p = \frac{x}{e^x - 1}.$$

Recall that $B_{2n+1} = 0$, $n \geq 1$. Define

$$S_j(x) = \frac{1}{j+1} \sum_{p=0}^j (-1)^p \binom{j+1}{p} B_p x^{j+1-p}, \quad j \geq 1. \quad (0.13)$$

Define also polynomials $F_l(u)$ by

$$F_l(u) = l u^l E_l(\frac{1}{u}) = \sum_{t|l} \mu(t) u^{l-\frac{l}{t}} = 1 - u^{l-1/p_1} - u^{l-1/p_2} + u^{l-1/p_1 p_2} + \dots + \mu(l) u^{l-1}, \quad (0.14)$$

where p_1 and p_2 are first prime factors of l . Notice one always has $F_l(0) = 1$.

Similarly to (0.10) one can consider the generating function $F_{m_1, \dots, m_r, d}^\pi(x_1, \dots, x_r, u)$ associated to the splitting (0.9).

Theorem 0.3. The generating function $F_{m_1, \dots, m_r, d}^\pi(x_1, \dots, x_r, u)$ is given by the formula

$$F_{m_1, \dots, m_r, d}^\pi(x_1, \dots, x_r, u) = \sum_{k, l, j \geq 1} \frac{\mu(k)}{kj} S_j \left(\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i^k) \right) \left(\frac{(-1)^{d-1} l u^{kl}}{F_l(u^k)} \right)^j - \sum_{k, l \geq 1} \sum_{i=1}^r \frac{\mu(k)}{k} (-1)^{m_i-1} E_l(x_i^k) \ln(F_l(u^k)),$$

where the polynomials E_l , F_l , S_j are respectively defined by (0.11), (0.14), (0.13).

The latter result essentially encodes the same information as the *supercharacter* of the symmetric group action on the positive arity components of $\mathbf{Mod}(L_\infty)$ described by Theorem 0.4 below, which is the third result of this paper.

Theorem 0.4. The supercharacter of the symmetric group action on the positive arity components $\{\mathbf{Mod}(L_\infty)(k)\}_{k \geq 1}$ of the modular envelop of L_∞ is described by the cycle index sum

$$Z_{\mathcal{X}(\mathbf{Mod}(L_\infty)(\bullet))}(w; p_1, p_2, p_3, \dots) = w \sum_{k, l, j \geq 1} \frac{\mu(k)}{kj} S_j \left(\frac{1}{l} \sum_{a|l} \mu\left(\frac{l}{a}\right) \frac{p_{ak}}{w^{ak}} \right) \left(\frac{l w^{kl}}{F_l(w^k)} \right)^j - w \sum_{k, l \geq 1} \frac{\mu(k)}{kl} \left(\sum_{a|l} \mu\left(\frac{l}{a}\right) \frac{p_{ak}}{w^{ak}} \right) \ln(F_l(w^k)),$$

where the variable w is responsible for the genus.

Outline of the paper.

- In Section 1 we prove Theorem 0.1, which presents a formula for the generating function of Euler characteristics of summands in the homological splitting (0.8). We end with Subsection 1.3, which gives formulas obtained from (0.12) by taking $m_i = 1$, $x_i = \pm 1$, $1 \leq i \leq r$.
- In Section 2 we prove three results: Theorem 0.3, Theorem 2.5 and Theorem 2.8. The first one computes the generating function of Euler characteristics of summands in the homotopical splitting (0.9). The key point is Lemma 2.1, which presents the generating function in homotopy as a sum of logarithms of the generating function in homology. The second result (respectively the third result) computes the generating function for the homology ranks of the summands in (0.9) of genus zero (respectively of genus one). Both from Theorem 2.5 and Theorem 2.8 one can see the exponential growth of the Betti numbers of the rational homotopy of $\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$, $r \geq 2$.
- In Section 3 we prove Theorem 0.4, which determines the supercharacter of the symmetric group action on the positive arity components of the modular envelop of L_∞ . This result is an interesting byproduct result for the theory of operads, and it is obtained from the computations we did in the previous section.
- In the appendix we produce, using the generating function from Theorem 0.3, tables of Euler characteristics of the summands in the homotopical splitting (0.9).

Acknowledgements: This work has been supported by Fonds de la Recherche Scientifique-FNRS (F.R.S.-FNRS), that the authors acknowledge. It has been also supported by the Kansas State University (KSU), where this paper was partially written during the stay of the first author, and which he thanks for hospitality.

1 Generating functions of Euler characteristics in homology

The goal of this section is to prove Theorem 0.1, which gives a formula for the generating function of the Euler characteristics of the summands in the homological splitting (0.10).

1.1 Proof of Theorem 0.1

In order to prove Theorem 0.1, we recall the definition of Koszul complexes computing (0.8). In short these complexes were obtained in [22] by taking the projective resolution of the source Ω -modules.

1.1.1 Koszul complexes

We start with a definition and a proposition.

Definition 1.1. Let $V = \{V(n)\}_{n \geq 0}$ and $W = \{W(n)\}_{n \geq 0}$ be two symmetric sequences. Define a new symmetric sequence $V \widehat{\otimes} W$ by

$$V \widehat{\otimes} W(n) = \bigoplus_{p+q=n} \text{Ind}_{\Sigma_p \times \Sigma_q}^{\Sigma_n} V(p) \otimes W(q).$$

In practice we need to deal with multigraded vector spaces. Besides the usual homological degree, they will have the Hodge multi-grading (s_1, \dots, s_r) and grading by complexity t . As usual when we take tensor product all the degrees get added.

Recalling the notation $\widehat{H}_*(-)$ from (0.1), we have the following result.

Proposition 1.2. [22, Proposition 4.5] For $d > 2\max\{m_i \mid 1 \leq i \leq r\} + 1$, there is a quasi-isomorphism

$$C_*(\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)) \otimes \mathbb{Q} \simeq \left(\bigoplus_{k \geq 0} \text{hom}_{\Sigma_k} \left(\bigotimes_{1 \leq i \leq r} \overline{H}_*(C(\bullet, \mathbb{R}^{m_i}), \mathbb{Q})(k), \widehat{H}_*(C(k, \mathbb{R}^d), \mathbb{Q}) \right), \partial \right). \quad (1.1)$$

Here $\overline{H}_*(-)$ is the Borel-Moore homology functor.

The differential ∂ here is defined similarly to the $r = 1$ case [2, Section 5]. We won't describe it explicitly here. However, in [22, Subsection 4.3] we explicitly describe the dual complex computing the cohomology of the space of links.

From now on let us assume that all $m_i > 1$. We can do so because the summands in (0.8) up to a regrading depend only on the parities of m_i , $i = 1 \dots r$, and that of d . The right hand side of (1.1) can be rewritten as

$$\bigoplus_{s_1, \dots, s_r, t \geq 0} \left(\bigoplus_{k \geq 0} \text{hom}_{\Sigma_k} \left(\bigoplus_{k=k_1+\dots+k_r} \left(\text{Ind}_{\Sigma_{\vec{k}}}^{\Sigma_k} \bigotimes_{i=1}^r \overline{H}_{s_i(m_i-1)+k_i}(C(k_i, \mathbb{R}^{m_i}), \mathbb{Q}) \right), \widehat{H}_{t(d-1)}(C(k, \mathbb{R}^d), \mathbb{Q}) \right), \partial \right),$$

where $\Sigma_{\vec{k}} = \Sigma_{k_1} \times \dots \times \Sigma_{k_r}$.

The s_1, \dots, s_r, t summand in the complex above computes exactly the corresponding summand in (0.8). Thus the Euler characteristic $\mathcal{X}_{s_1 \dots s_r t}$ used in (0.10) can be defined as

$$\mathcal{X}_{s_1 \dots s_r t} = \sum_{k \geq 0} (-1)^{t(d-1) - \sum_{i=1}^r s_i(m_i-1) - k} \dim X_k, \quad (1.2)$$

where

$$X_k = \text{hom}_{\Sigma_k} \left(\bigoplus_{k=k_1+\dots+k_r} \left(\text{Ind}_{\Sigma_{\vec{k}}}^{\Sigma_k} \bigotimes_{i=1}^r \overline{H}_{s_i(m_i-1)+k_i}(C(k_i, \mathbb{R}^{m_i}), \mathbb{Q}) \right), \widehat{H}_{t(d-1)}(C(k, \mathbb{R}^d), \mathbb{Q}) \right).$$

1.1.2 Cycle index sum and proof of Theorem 0.1

Before starting the proof of Theorem 0.1 we will state two lemmas. Let us first introduce some notation. For a permutation $\sigma \in \Sigma_k$, let $j_l(\sigma)$ denote the number of its cycles of length l . Let $\text{tr}(-)$ denote the trace function from the space of matrices to the ground field.

Definition 1.3. Let $\{p_1, p_2, \dots\}$ be a family of commuting variables.

- For a representation $\rho^V: \Sigma_k \rightarrow GL(V)$ of the symmetric group Σ_k , the cycle index sum of V , denoted $Z_V(p_1, p_2, \dots)$, is defined by

$$Z_V(p_1, p_2, \dots) = \frac{1}{|\Sigma_k|} \sum_{\sigma \in \Sigma_k} \text{tr}(\rho^V(\sigma)) \prod_l p_l^{j_l(\sigma)}. \quad (1.3)$$

- If $V = \{V(k)\}_{k \geq 0}$ is a symmetric sequence, we define $Z_V(p_1, p_2, \dots) = \sum_{k \geq 0} Z_{V(k)}(p_1, p_2, \dots)$.

The vector spaces in the symmetric sequences below will be always multigraded. The trace in (1.3) will be a graded trace, i.e. it will be a generating function of traces on each component. For example, if

$$V = \bigoplus_{i, s_1, \dots, s_r} V_{i, s_1, \dots, s_r},$$

where i is the homological degree, and s_1, \dots, s_r is the Hodge multigrading, then

$$\mathrm{tr}(\rho^V(\sigma)) = \sum_{i, s_1, \dots, s_r} \mathrm{tr}(\rho^{V_{i, s_1, \dots, s_r}}(\sigma)) z^i x_1^{s_1} \dots x_r^{s_r}.$$

With such definition, the cycle index sum Z_V of a symmetric sequence will also depend on z, x_1, \dots, x_r , and also possibly on u , where the last variable will be responsible for the grading by complexity.

Given two families $\{p_1, p_2, \dots\}$ and $\{b_1, b_2, \dots\}$ of commuting variables, we will write $Z_V(p_l \leftarrow b_l; l \in \mathbb{N})$ for $Z_V(b_1, b_2, \dots)$. For a function $f = f(p_1, p_2, \dots)$ on variables p_1, p_2, \dots , the notation $\left\{ Z_V\left(\frac{\partial}{\partial p_l}; l \in \mathbb{N}\right) f(p_1, p_2, \dots) \right\} \Big|_{p_l=0}$ means that we apply the differential operator $Z_V\left(\frac{\partial}{\partial p_l}; l \in \mathbb{N}\right)$ to f , and at the end we take $p_l = 0$ for all $l \geq 0$.

The following lemma is well known, see for example [23, Corollary 15.5].

Lemma 1.4. *Let $V = \oplus_i V_i$ and $W = \oplus_j W_j$ be two graded vector spaces admitting an action of the symmetric group Σ_k on each of them. Consider the series*

$$\dim \mathrm{hom}_{\Sigma_k}(V, W) = \sum_{i, j} \dim \mathrm{hom}_{\Sigma_k}(V_i, W_j) z^{j-i},$$

$$Z_V(z; p_1, p_2, \dots) = \sum_i Z_{V_i}(p_1, p_2, \dots) z^i \quad \text{and} \quad Z_W(z; p_1, p_2, \dots) = \sum_j Z_{W_j}(p_1, p_2, \dots) z^j.$$

Then

$$\dim \mathrm{hom}_{\Sigma_k}(V, W) = \left\{ Z_V\left(\frac{1}{z}; p_l \leftarrow \frac{\partial}{\partial p_l}, l \in \mathbb{N}\right) Z_W(z; p_l \leftarrow l p_l, l \in \mathbb{N}) \right\} \Big|_{p_l=0} = \left\{ Z_V\left(\frac{1}{z}; p_l \leftarrow l \frac{\partial}{\partial p_l}, l \in \mathbb{N}\right) Z_W(z; p_1, p_2, \dots) \right\} \Big|_{p_l=0}. \quad (1.4)$$

For the application below V will be in addition Hodge multigraded and W will be in addition graded by complexity. This will add up variables in the expression.

The second result we need to prove Theorem 0.1 is the following well known lemma, see for example [23, Proposition 15.3] or [8].

Lemma 1.5. *Let $V = \{V(k)\}_{k \geq 0}$ and $W = \{W(k)\}_{k \geq 0}$ be two finite symmetric sequences (that is, for all $k \geq 0$, $V(k)$ and $W(k)$ are of finite dimensions). Then*

$$Z_{V \hat{\otimes} W}(p_1, p_2, \dots) = Z_V(p_1, p_2, \dots) Z_W(p_1, p_2, \dots).$$

Again we will be using a multigraded version of this lemma.

We are now ready to prove the main result of this section.

Proof of Theorem 0.1. In all this proof $\Sigma_{\bar{k}} = \Sigma_{k_1} \times \dots \times \Sigma_{k_r}$. Consider the following generating function

$$\Psi_{m_1 \dots m_r, d}(x_1, \dots, x_r, u, z) = \sum_{s_1, \dots, s_r, k} \dim \left(\mathrm{hom}_{\Sigma_{\bar{k}}} \left(\bigoplus_{k=k_1+\dots+k_r} \left(\mathrm{Ind}_{\Sigma_{\bar{k}}}^{\Sigma_k} \otimes_{i=1}^r \bar{H}_{s_i(m_i-1)+k_i}(C(k_i, \mathbb{R}^{m_i}), \mathbb{Q}) \right), \hat{H}_{t(d-1)}(C(k, \mathbb{R}^d), \mathbb{Q}) \right) \right) \alpha,$$

where $\alpha = x_1^{s_1} \dots x_r^{s_r} u^t z^{t(d-1) - \sum_{i=1}^r s_i(m_i-1) - k}$, and define two symmetric sequences V and W by

$$V(k) = \bigoplus_{k=k_1+\dots+k_r} \left(\mathrm{Ind}_{\Sigma_{\bar{k}}}^{\Sigma_k} \otimes_{i=1}^r \bar{H}_*(C(k_i, \mathbb{R}^{m_i}), \mathbb{Q}) \right) \quad \text{and} \quad W(k) = \hat{H}_*(C(k, \mathbb{R}^d), \mathbb{Q}).$$

Then, by Lemma 1.4, it is not difficult to see that the generating function $\Psi_{m_1 \dots m_r, d}(x_1, \dots, x_r, u, z)$ is

$$\Psi_{m_1 \dots m_r, d}(x_1, \dots, x_r, u, z) = \left\{ Z_V\left(\frac{1}{z}, x_1, \dots, x_r; p_l \leftarrow \frac{\partial}{\partial p_l}, l \geq 1\right) Z_W(z, u; p_l \leftarrow l p_l, l \geq 1) \right\} \Big|_{p_l=0}.$$

Since (recalling the definition of $E_l(-)$ from the introduction, just before Theorem 0.1)

- $Z_W(z, u; p_l \leftarrow l p_l, l \geq 1) = \prod_{l=1}^{+\infty} e^{-p_l} (1 + (-1)^d l ((-z)^{d-1} u)^l p_l)^{(-1)^d E_l(\frac{1}{(-z)^{d-1} u})}$ by [2, Proposition 6.4],
- $Z_V(\frac{1}{z}, x_1, \dots, x_r; p_l \leftarrow \frac{\partial}{\partial p_l}, l \geq 1) = \prod_{i=1}^r Z_{\overline{H}_*(C(\bullet, \mathbb{R}^{m_i}), \mathbb{Q})}(\frac{1}{z}, x_i; p_l \leftarrow \frac{\partial}{\partial p_l}, l \geq 1)$ by Lemma 1.5 and the fact that $V = \widehat{\bigotimes}_{i=1}^r \overline{H}_*(C(\bullet, \mathbb{R}^{m_i}), \mathbb{Q})$,

and since

$$Z_{\overline{H}_*(C(\bullet, \mathbb{R}^{m_i}), \mathbb{Q})}(\frac{1}{z}, x_i; p_l \leftarrow \frac{\partial}{\partial p_l}, l \geq 1) = \prod_{l=1}^{+\infty} \left(1 + (-\frac{1}{z})^l \frac{\partial}{\partial p_l}\right)^{(-1)^{m_i} E_l(\frac{x_i}{(-z)^{m_i-1}})} \text{ by [2, Proposition 6.5],}$$

it follows that

$$\begin{aligned} & \Psi_{m_1 \dots m_r, d}(x_1, \dots, x_r, u, z) \\ &= \left\{ \prod_{i=1}^r \prod_{l=1}^{+\infty} \left(1 + (-\frac{1}{z})^l \frac{\partial}{\partial p_l}\right)^{(-1)^{m_i} E_l(\frac{x_i}{(-z)^{m_i-1}})} e^{-p_l} (1 + (-1)^d l ((-z)^{d-1} u)^l p_l)^{(-1)^d E_l(\frac{1}{(-z)^{d-1} u})} \right\} \Big|_{p_l=0} \\ &= \left\{ \prod_{l=1}^{+\infty} \left(1 + (-\frac{1}{z})^l \frac{\partial}{\partial p_l}\right)^{\sum_{i=1}^r (-1)^{m_i} E_l(\frac{x_i}{(-z)^{m_i-1}})} e^{-p_l} (1 + (-1)^d l ((-z)^{d-1} u)^l p_l)^{(-1)^d E_l(\frac{1}{(-z)^{d-1} u})} \right\} \Big|_{p_l=0}. \end{aligned}$$

Notice that the l -th factor uses p_l and not any other p_i , $i \neq l$, which is then taken to be zero. For this reason in the formula below we replace each p_l by a . By looking at (0.10), (1.2), and the definition of $\Psi_{m_1 \dots m_r, d}(x_1, \dots, x_r, u, z)$, we have the equality

$$F_{m_1 \dots m_r, d}^H(x_1, \dots, x_r, u) = \Psi_{m_1 \dots m_r, d}(x_1, \dots, x_r, u, -1),$$

and this implies

$$\begin{aligned} F_{m_1 \dots m_r, d}^H(x_1, \dots, x_r, u) &= \left\{ \prod_{l=1}^{+\infty} \left(1 + \frac{\partial}{\partial a}\right)^{\sum_{i=1}^r (-1)^{m_i} E_l(x_i)} e^{-a} (1 + (-1)^d l u^l a)^{(-1)^d E_l(\frac{1}{u})} \right\} \Big|_{a=0} \\ &= \left\{ \prod_{l=1}^{+\infty} \left(1 + \frac{\partial}{\partial a}\right)^{-\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i)} e^{-a} (1 + (-1)^d l u^l a)^{(-1)^d E_l(\frac{1}{u})} \right\} \Big|_{a=0} \\ &= \prod_{l=1}^{+\infty} \frac{\Gamma((-1)^{d-1} E_l(\frac{1}{u}) - \sum_{i=1}^r (-1)^{m_i-1} E_l(x_i))}{((-1)^{d-1} l u^l)^{\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i)} \Gamma((-1)^{d-1} E_l(\frac{1}{u}))}. \end{aligned}$$

The last equality follows from the identity

$$\left(1 + \frac{\partial}{\partial a}\right)^{-X} e^{-a} (1 + (-1)^d l u^l a)^{(-1)^d E_l(\frac{1}{u})} \Big|_{a=0} = \frac{\Gamma((-1)^{d-1} E_l(\frac{1}{u}) - X)}{((-1)^{d-1} l u^l)^X \Gamma((-1)^{d-1} E_l(\frac{1}{u}))},$$

whose proof is identical to that of [23, Proposition 15.7].

We thus obtain the desired result. \square

1.2 Understanding generating function of the Euler characteristics of the homology splitting

The formula (0.12) might appear confusing at first as it is defined in terms of asymptotic expansions that still need to be deciphered. Let us first fix some notation. For two variables x, u , let $\Gamma(x, u)$ denote the asymptotic expansion of $\frac{\Gamma(\frac{1}{u}-x)}{u^x \Gamma(\frac{1}{u})}$ when u goes to $+0$. For instance, in the case $x = n$ is a positive integer, one has

$$\Gamma(n, u) = \frac{1}{(1-u)(1-2u) \cdots (1-nu)}. \quad (1.5)$$

In the case x is a negative integer: $x = -n$, one has

$$\Gamma(-n, u) = (1+u)(1+2u) \cdots (1+(n-1)u). \quad (1.6)$$

(In particular $\Gamma(-1, u) = \Gamma(0, u) = 1$.) The series $\Gamma(x, u)$ can be seen as a generating function in u of a sequence of polynomials in x , see [23, Subsection 14.1]. It has rather bad convergency properties: for any $x \in \mathbb{C} \setminus \mathbb{Z}$ its radius of convergence in u is zero [23, Proposition 14.5].

Proposition 1.6.

$$F_{m_1 \dots m_r, d}^H(x_1, \dots, x_r, u) = \frac{\prod_{l \geq 1} \Gamma\left(\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i), (-1)^{d-1} \frac{lu^l}{F_l(u)}\right)}{\prod_{l \geq 1} (F_l(u))^{\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i)}}, \quad (1.7)$$

where polynomials $F_l(u)$ are defined in (0.14).

Proof. This is deduced from Theorem 0.1 and from the formula

$$\frac{\Gamma((-1)^{d-1} E_l(\frac{1}{u}) - X)}{((-1)^{d-1} lu^l)^X \Gamma((-1)^{d-1} E_l(\frac{1}{u}))} = \frac{\Gamma\left(X, (-1)^{d-1} \frac{lu^l}{F_l(u)}\right)}{F_l(u)^X},$$

see [23, Lemma 14.6] for similar computations for the case of long knots. \square

1.3 Some special cases of the generating function in homology

The generating function $F_{m_1 \dots m_r, d}^H(x_1, \dots, x_r, u)$ from (0.12) can be used to estimate the ranks of the homology and homotopy groups of $\text{Emb}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$. However, the formula itself is hard to use to estimate the asymptotics of the growth since it has infinitely many rather complicated factors. However, sometimes it is possible to choose specific values for x_i that would kill all but finitely many factors. This has been done in the case of long knots $r = 1$, $m_1 = 1$ in [23], where choosing $x_1 = -1$ allows one to prove the exponential growth of the ranks of the rational homology and homotopy groups of the space of long knots. Notice that choosing $x_1 = 1$ (all $x_i = 1$ in the general case) corresponds to forgetting the Hodge splitting. In the case of long knots the latter choice does not prove exponential growth, see [15, 23] or computations below, which showed that the Hodge splitting was essential.

In this subsection we produce similar computations for the spaces of classical string links: all $m_i = 1$, $i \in \{1, \dots, r\}$. Recall that in [15] it has been shown that the homology ranks of $\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^1, \mathbb{R}^d)$ grow exponentially for $r \geq 2$ without using the Hodge splitting. The computations that we produce below are rather disappointing: we compare the choice all $x_i = -1$ versus all $x_i = 1$ and see that only for $r \leq 2$ the first choice may give a better estimate.²

²As we said earlier it does give a better estimate for $r = 1$. In case $r = 2$ and d odd it also gives a better estimate, but the case of $r = 2$ and even d is more subtle as both generating functions happen to have the same radius of convergence.

- Assume d odd, and consider the function $F_{\text{odd}}(1, u)$ defined by $F_{\text{odd}}(1, u) = F_{1\dots 1, d}^H(1, \dots, 1, u)$. We want to compute $F_{\text{odd}}(1, u)$. By substituting m_i by 1, and x_i by 1, $1 \leq i \leq r$, in the formula (1.7), and by noticing that $E_l(1) = \begin{cases} 1 & \text{if } l = 1 \\ 0 & \text{if } l \geq 2 \end{cases}$, we obtain

$$F_{\text{odd}}(1, u) = \Gamma(r, u) = \frac{1}{(1-u)(1-2u)\cdots(1-ru)},$$

where the last equality comes from (1.5). The radius of convergence in this case is $\frac{1}{r}$ which implies exponential growth of rational homology ranks in case $r \geq 2$.

- For d even, similar computations give

$$F_{\text{even}}(1, u) = F_{1\dots 1, d}^H(1, \dots, 1, u) = \frac{1}{(1+u)(1+2u)\cdots(1+ru)}. \quad (1.8)$$

The radius of convergence is $\frac{1}{r}$.

- Assume d odd, and consider the function $F_{\text{odd}}(-1, u)$ defined by $F_{\text{odd}}(-1, u) = F_{1\dots 1, d}^H(-1, \dots, -1, u)$. By substituting m_i by 1, and x_i by -1 , $1 \leq i \leq r$, in the formula (1.7), and by noticing that $E_l(-1) = \begin{cases} (-1)^l & \text{if } l \in \{1, 2\} \\ 0 & \text{if } l \geq 3 \end{cases}$, we obtain

$$F_{\text{odd}}(-1, u) = \Gamma(-r, u) \times \frac{\Gamma\left(r, \frac{2u^2}{F_2(u)}\right)}{F_2(u)^2}. \quad (1.9)$$

Since $F_2(u) = 1 - u$ and applying (1.5) and (1.6), the formula (1.9) produces

$$F_{\text{odd}}(-1, u) = \frac{(1+u)(1+2u)\cdots(1+(r-1)u)}{(1-u-2u^2)(1-u-4u^2)\cdots(1-u-2ru^2)} \quad (1.10)$$

It is easy to see that the smallest root (in absolute value) of the denominator of (1.10) is $\frac{-1+\sqrt{1+8r}}{4r}$. Therefore the radius of convergence is equal to $\frac{-1+\sqrt{1+8r}}{4r}$. This gives a better estimate of the homology ranks growth for $r \leq 2$. In case $r \geq 4$, the produced estimate is weaker than the one obtained from $F_{\text{odd}}(1, u)$.

- For d even, similar computations give

$$F_{\text{even}}(-1, u) = \frac{(1-u)(1-2u)\cdots(1-(r-1)u)}{(1-u+2u^2)(1-u+4u^2)\cdots(1-u+2ru^2)} \quad (1.11)$$

As before, the radius of convergence of (1.11) is equal to $\left| \frac{1+\sqrt{1-8r}}{4r} \right| = \frac{1}{\sqrt{2r}}$. In particular for $r = 1$ this implies exponential growth of the homology ranks. In case $r \geq 3$, the produced estimate is weaker than the one obtained from $F_{\text{even}}(1, u)$.

2 Generating functions of Euler characteristics in homotopy

The goal of this section is to prove Theorem 0.3, which computes the generating function of Euler characteristics of summands in the homotopical splitting (0.9). Another goal is to compute, using a different complex, the generating function of the ranks of the summands in (0.9) of genus zero and one.³

³Genus g is defined as the complexity minus total Hodge degree plus one $g = t - (s_1 + \dots + s_r) + 1$.

2.1 The generating function for $\mathbb{Q} \otimes \pi_*(\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d))$

In Section 1 we have computed the generating function $F_{m_1 \dots m_r u}^H(x_1, \dots, x_r, u)$ of Euler characteristics of summands in the homological splitting of $H_*(\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d), \mathbb{Q})$. The aim of this section is to compute the generating function $F_{m_1 \dots m_r u}^\pi(x_1, \dots, x_r, u)$ of the Euler characteristics of summands, but now in the homotopy $\mathbb{Q} \otimes \pi_*(\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d))$. Proposition 1.6 will be a key ingredient in our computations. If there is no confusion, the first generating function will be denoted by $F^H(x_1, \dots, x_r, u)$, and the second one just by $F^\pi(x_1, \dots, x_r, u)$. Recalling the equation (0.9) from the introduction, for a sequence $s_1, \dots, s_r, t \geq 0$, let $\mathcal{X}_{s_1 \dots s_r t}^\pi$ denote the Euler characteristic of the summand $\text{hRmod}(Q_{s_1 \dots s_r}^{m_1 \dots m_r}, \mathbb{Q} \otimes \widehat{\pi}_{t(d-2)+1}(C(\bullet, \mathbb{R}^d)))$. We will also write \bar{s} for s_1, \dots, s_r , and $\bar{x}^{\bar{s}}$ for $\prod_{i=1}^r x_i^{s_i}$. The generating function we look at here is defined by

$$F_{m_1 \dots m_r d}^\pi(x_1, \dots, x_r, u) = \sum_{\bar{s}, t} \mathcal{X}_{\bar{s}, t}^\pi \bar{x}^{\bar{s}} u^t. \quad (2.1)$$

In order to compute $F_{m_1 \dots m_r d}^\pi(x_1, \dots, x_r, u)$, we need the following lemma, which connects $F^H(x_1, \dots, x_r, u)$ and $F^\pi(x_1, \dots, x_r, u)$.

Lemma 2.1. *We have*

$$F_{m_1 \dots m_r d}^\pi(x_1, \dots, x_r, u) = \sum_{l=1}^{+\infty} \frac{\mu(l)}{l} \ln(F^H(x_i \leftarrow x_i^l, u \leftarrow u^l)). \quad (2.2)$$

Here $\mu(-)$ is the standard Möbius function. The notation $F^H(x_i \leftarrow x_i^l, u \leftarrow u^l)$ means that in the expression $F^H(x_1, \dots, x_r, u)$ the variable u is replaced by u^l , and x_i is replaced by x_i^l , $1 \leq i \leq r$.

Proof. The plan of the proof is to compute the right hand side of (2.2), and compare the obtained result with the right hand side of (2.1). From [23, Lemma 16.1], it is straightforward to get the following

$$F^H(x_1, \dots, x_r, u) = \prod_{\bar{s}, t} \frac{1}{(1 - \bar{x}^{\bar{s}} u^t)^{\mathcal{X}_{\bar{s}, t}^\pi}}, \quad (2.3)$$

which gives (by replacing u by u^l and x_i by x_i^l)

$$F^H(x_i \leftarrow x_i^l, u \leftarrow u^l) = \prod_{\bar{s}, t} \frac{1}{(1 - (\bar{x}^{\bar{s}} u^t)^l)^{\mathcal{X}_{\bar{s}, t}^\pi}}. \quad (2.4)$$

Taking now the logarithm of (2.4), and using the well known series expansion $\ln(1 - x) = -\sum_{p \geq 1} \frac{x^p}{p}$, we obtain

$$\ln(F^H(x_i \leftarrow x_i^l, u \leftarrow u^l)) = \sum_{\bar{s}, t} \sum_{p \geq 1} \mathcal{X}_{\bar{s}, t}^\pi \frac{(\bar{x}^{\bar{s}} u^t)^{pl}}{p}. \quad (2.5)$$

By putting (2.5) in the right hand side of (2.2), we get

$$\begin{aligned} \sum_{l=1}^{+\infty} \frac{\mu(l)}{l} \ln(F^H(x_i \leftarrow x_i^l, u \leftarrow u^l)) &= \sum_{l \geq 1} \sum_{\bar{s}, t} \sum_{p \geq 1} \mathcal{X}_{\bar{s}, t}^\pi \mu(l) \frac{(\bar{x}^{\bar{s}} u^t)^{pl}}{pl} \\ &= \sum_{\bar{s}, t} \sum_{q \geq 1} \mathcal{X}_{\bar{s}, t}^\pi \frac{1}{q} (\bar{x}^{\bar{s}} u^t)^q \left(\sum_{l|q} \mu(l) \right). \end{aligned}$$

Since $\sum_{l|q} \mu(l) = \begin{cases} 1 & \text{if } q = 1 \\ 0 & \text{if } q \geq 2 \end{cases}$, it follows that

$$\begin{aligned} \sum_{l=1}^{+\infty} \frac{\mu(l)}{l} \ln(F^H(x_i \leftarrow x_i^l, u \leftarrow u^l)) &= \sum_{\bar{s}, t} \mathcal{X}_{\bar{s}t}^{\pi} \bar{x}^{\bar{s}} u^t \\ &= F^{\pi}(x_1, \dots, x_r, u) \text{ by (2.1).} \end{aligned}$$

□

The last thing we need is an explicit expansion of $\ln(\Gamma(x, u))$, where $\Gamma(x, u)$ is defined at the beginning of subsection 1.2. Recalling (0.13) from the introduction, one can notice that when $x = n$ is a positive integer, we have $S_j(n) = \sum_{p=1}^n p^j, j \geq 1$.

Lemma 2.2.

$$\ln(\Gamma(x, u)) = \sum_{j \geq 1} S_j(x) \frac{u^j}{j}.$$

Proof. This identity easily follows from (1.5) when $x = n$ is an integer. It also appears in the proof of [23, Proposition 14.5]. □

Now we are ready to prove Theorem 0.3, which is the main result of this subsection.

Proof of Theorem 0.3. The proof is a direct application of Proposition 1.6 and Lemmas 2.1-2.2. □

2.2 The generating function in homotopy for genus ≤ 1

In [22, Subsection 2.1] we explicitly described a complex $\mathcal{E}_{\pi}^{m_1, \dots, m_r, d}$ of hairy graphs that computes the rational homotopy of the space $\text{Emb}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ of high dimensional string links. In short, this complex is obtained in [22] by taking the injective resolution of the target Ω -modules in (0.9). The graph-complex $\mathcal{E}_{\pi}^{m_1, \dots, m_r, d}$ is a differential graded vector space spanned by so called *hairy graphs*, i.e. connected graphs having a finite non-empty set of external vertices (of valence 1 and called *hairs*), a finite set of non-labeled internal vertices (of valence ≥ 3). The edges are oriented. One allows both multiple edges and tadpoles. The external vertices are colored with $\{1, \dots, r\}$ as the set of colours. As a part of the data, each graph comes with the ordering of its *orientation set* that consists of the following elements: edges (of degree $d-1$), internal vertices (of degree $-d$), external vertices (of degree $-m_i$ if it is colored by i). Changing orientation of an edge in a graph produces sign $(-1)^d$. Changing the order of the orientation set produces the Koszul sign of permutation that takes into account the degree of the elements. The differential is defined as the sum of expansion of internal vertices.

For a hairy graph $G \in \mathcal{E}_{\pi}^{m_1, \dots, m_r, d}$, its homological degree is the sum of the degrees of the elements in its orientation set. Its corresponding *Hodge multi-degree* is the tuple (s_1, \dots, s_r) , where s_i is the number of external vertices colored by i . Its *complexity* is the first Betti number of the graph obtained from G by gluing together all its external vertices. For hairy graphs it is also natural to define *genus* g as their first Betti number.

As an example of the complexity, the graph in Figure 1 is of complexity 2, while the complexity is three in Figure 2 (and five in Figure 3). Concerning the genus, intuitively, a graph is of genus g if it contains g “loops”. See Figure 1, Figure 2 and Figure 3 for some examples of graphs of genus 0, 1 and 2 respectively.

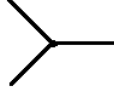


Figure 1: Graph of genus 0

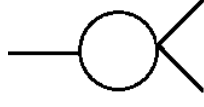


Figure 2: Graph of genus 1

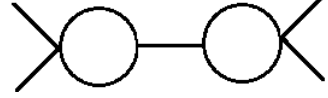


Figure 3: Graph of genus 2

The graph-complex $\mathcal{E}_\pi^{m_1, \dots, m_r, d}$ can be split as follows. For $g \geq 0$, let $\mathcal{E}_{g\pi}^{m_1, \dots, m_r, d}$ denote the subcomplex of $\mathcal{E}_\pi^{m_1, \dots, m_r, d}$ generated by graphs of genus g . For integers s_1, \dots, s_r, t , let $\mathcal{E}_{g\pi s_1, \dots, s_r, t}^{m_1, \dots, m_r, d}$ be the subcomplex of $\mathcal{E}_{g\pi}^{m_1, \dots, m_r, d}$ generated by graphs of Hodge multi-degree (s_1, \dots, s_r) and complexity t . All these subcomplexes are well defined since the differential in $\mathcal{E}_\pi^{m_1, \dots, m_r, d}$ preserves the Hodge multi-degree, the complexity, and the genus. In particular, one has the splitting

$$\mathcal{E}_\pi^{m_1, \dots, m_r, d} = \bigoplus_{g \geq 0} \mathcal{E}_{g\pi}^{m_1, \dots, m_r, d}. \quad (2.6)$$

Definition 2.3. *Let $g \geq 0$.*

- *The Hodge splitting of the complex $\mathcal{E}_{g\pi}^{m_1, \dots, m_r, d}$ is the splitting*

$$\mathcal{E}_{g\pi}^{m_1, \dots, m_r, d} = \bigoplus_{s_1, \dots, s_r, t \geq 0} \mathcal{E}_{g\pi s_1, \dots, s_r, t}^{m_1, \dots, m_r, d}. \quad (2.7)$$

- *The generating function associated to (2.7), and denoted $F_{m_1, \dots, m_r, d}^{g\pi}(x_1, \dots, x_r, u)$, is defined as*

$$F_{m_1, \dots, m_r, d}^{g\pi}(x_1, \dots, x_r, u) = \sum_{s_1, \dots, s_r, t \geq 0} \mathcal{X}_{g\pi s_1, \dots, s_r, t}^\pi x_1^{s_1} \cdots x_r^{s_r} u^t, \quad (2.8)$$

where $\mathcal{X}_{g\pi s_1, \dots, s_r, t}^\pi$ is the Euler characteristic of $\mathcal{E}_{g\pi s_1, \dots, s_r, t}^{m_1, \dots, m_r, d}$.

It is easy to see the following relation between the genus and the complexity of a graph.

$$\text{genus} = \text{complexity} - \text{number of external vertices} + 1. \quad (2.9)$$

Thanks to (2.9), we can easily express the generating function of the Euler characteristics that takes into account the grading g as well:

$$F^\pi(x_1, \dots, x_r, u, w) = w F^\pi\left(\frac{x_1}{w}, \dots, \frac{x_r}{w}, uw\right), \quad (2.10)$$

where w is the variable responsible for the genus, and the right-hand side uses the generating function defined earlier (2.1). (Abusing notation we denote this function by F as well). Applying the result from Theorem 0.3 one can get an explicit formula for $F^\pi(x_1, \dots, x_r, u, w)$. However, even though this formula is very simple for explicit computer calculations it's not at all obvious that it produces zero for the negative genus. Indeed, there will be summands in which w appears with negative exponent, that must somehow cancel out with each other.

The aim of this subsection is to compute the generating function from (2.8) with $g = 0$ and 1 using hairy graph-complexes, which is done by Theorems 2.5 and 2.8 below. We need first to define a symmetric sequence of graph-complexes $\{M(P_d^k)\}_{k \geq 1}$ that will be used in our computations.

2.2.1 Graph-complexes $M(P_d^k)$

Recall that d denotes the dimension of the ambient space. For $k \geq 0$, $M(P_d^k)$ is the complex of graphs defined essentially in the same way as the hairy graph-complex with the only difference that its graphs have exactly k external vertices labeled bijectively by $1, \dots, k$. Also we exclude external vertices from the orientation set of such graphs. Thus the orientation set of any graph $G \in M(P_d^k)$ is the union of the set E_G of edges of G (each edge being of degree $d-1$) and the set I_G of its internal vertices (each internal vertex being of degree $-d$). So that the total degree of G is $(d-1)|E_G| - d|I_G|$. The differential on $M(P_D^k)$ is defined in the same way as the sum of expansions of internal vertices.

Since the differential preserves the genus (first Betti number of a graph), each of $M(P_d^k)$ can be split into a direct sum by genus g . That is, one has

$$M(P_d^k) = \bigoplus_{g \geq 0} M_g(P_d^k).$$

Let V be the multi-graded r -dimensional vector space of colors⁴, whose basis is the set $\{v_1, \dots, v_r\}$, with each v_i being of degree m_i and of Hodge multi-degree $(\underbrace{0 \dots 0}_{i-1}, 1, \underbrace{0 \dots 0}_{r-i})$. Consider the symmetric sequence

$V^{\otimes \bullet} = \{V^{\otimes n}\}_{n \geq 0}$. One has an isomorphism

$$\mathcal{E}_\pi^{m_1, \dots, m_r, d} \cong \text{hom}_\Sigma(V^{\otimes \bullet}, M(P_d^\bullet)). \quad (2.11)$$

This implies

$$\mathcal{E}_{g\pi}^{m_1, \dots, m_r, d} \cong \text{hom}_\Sigma(V^{\otimes \bullet}, M_g(P_d^\bullet)). \quad (2.12)$$

Therefore the homology of each summand $\mathcal{E}_{g\pi}^{m_1, \dots, m_r, d}$ in (2.6) is completely determined by the symmetric sequence of the homology of $M_g(P_d^\bullet)$. It turns out that for $g = 0$ and 1 , the corresponding symmetric sequences can be easily described. Thus the plan to get Theorem 2.5 and Theorem 2.8 will be to compute the cycle index sums $Z_{H_*M_0(P_d^\bullet)}$, $Z_{H_*M_1(P_d^\bullet)}$ first, and then apply (3.6). The latter equation is a general formula that we obtain in the next section by first computing the cycle index sum of $V^{\otimes \bullet}$ (3.3) and then applying (1.4). We mention also at this point that for every given k both groups $H_*M_0(P_d^k)$ and $H_*M_1(P_d^k)$ are concentrated in a single homological degree. Thus computation of the homology ranks or of Euler characteristics carry essentially the same information.

2.2.2 Genus zero

By (2.12) the complex $\mathcal{E}_{0\pi}$ of colored hairy graphs of genus zero has the form (take $g = 0$)

$$\mathcal{E}_{0\pi} \cong \text{hom}_\Sigma(V^{\otimes \bullet}, M_0(P_d^\bullet)). \quad (2.13)$$

On the other hand the complexes $M_0(P_d^\bullet) = \{M_0(P_d^k)\}_{k \geq 1}$ are complexes of trees, whose homology is well known to be isomorphic up to a shift of degrees to the components of the (cyclic) Lie operad [24]. More precisely one has an isomorphism of Σ_k -modules

$$H_*(M_0(P_d^k)) \cong_{\Sigma_k} \Sigma^{k(d-2)-d+3} \text{Lie}(k-1) \otimes (\text{sign})^{\otimes d}. \quad (2.14)$$

The homology is concentrated in the smallest possible degree of uni-trivalent trees. Such a tree with k external vertices must have $k-2$ internal vertices and $2k-3$ edges. Thus its degree is $(2k-3)(d-1) - (k-2)d =$

⁴A color for us is a component of links in $\overline{\text{Emb}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)}$

$k(d-2)-d+3$. Our hairy graph-complex specialized to the gradings when $d=3$ and all $m_i=1$, contains in the bottom degree homology the space of univalent graphs modulo AS and IXX relations, which encodes the finite type invariants of string links in \mathbb{R}^3 [3]. The tree-part of this space is well studied [11, 18, 12, 20]. We failed to find in the literature formulas similar to those given by Theorems 2.4-2.5, but they could be easily derived from the results of aforementioned papers and are known to specialists.

Let $\mathbf{1}(\bullet)$ be the symmetric sequence defined by $\mathbf{1}(1) = \mathbb{Q}$, and $\mathbf{1}(n) = 0$ for all $n \neq 1$. One has

$$\mathrm{Lie}(n-1) \cong_{\Sigma_n} \mathrm{Ind}_{\Sigma_1 \times \Sigma_{n-1}}^{\Sigma_n} (\mathbf{1}(1) \otimes \mathrm{Lie}(n-1)) - \mathrm{Lie}(n), \quad n \geq 2.$$

Rationally this was proved in [18, 12] and integrally in [6]. Therefore $\mathrm{Lie}(\bullet-1) \cong_{\Sigma} \mathbf{1}(\bullet) \hat{\otimes} \mathrm{Lie}(\bullet) - \mathrm{Lie}(\bullet) + \mathbf{1}(\bullet)$. We add $\mathbf{1}(\bullet)$ to compensate subtraction of $\mathrm{Lie}(1)$, i.e. to have zero in arity one. This implies that (since $Z_{\mathbf{1}(\bullet)}(p_1, p_2, \dots) = p_1$)

$$Z_{\mathrm{Lie}(\bullet-1)} = (p_1 - 1)Z_{\mathrm{Lie}(\bullet)} + p_1.$$

Using now the following well known result (see for example [4]; another short and elegant proof is given in [7, Section 5.1])

$$Z_{\mathrm{Lie}(\bullet)}(p_1, p_2, \dots) = \sum_{l=1}^{+\infty} \frac{-\mu(l) \ln(1-p_l)}{l},$$

we get

$$Z_{\mathrm{Lie}(\bullet-1)}(p_1, p_2, \dots) = (1-p_1) \sum_{l=1}^{+\infty} \frac{\mu(l) \ln(1-p_l)}{l} + p_1. \quad (2.15)$$

From (2.14) and the fact that for genus zero graphs the complexity is the number of external vertices minus one, we get

$$Z_{H_* M_0(P_d^\bullet)}(u, z; p_1, p_2, \dots) = \frac{1}{z^{d-3}u} Z_{\mathrm{Lie}(\bullet-1)} \left(p_l \leftarrow (-1)^{(l-1)d} (z^{d-2}u)^l p_l, l \in \mathbb{N} \right). \quad (2.16)$$

To recall, the *Hodge splitting* is defined in (2.7).

Theorem 2.4. *The generating function of the dimensions and of the Hodge splitting of the complex $\mathcal{E}_{0\pi}^{m_1, \dots, m_r, d}$ of hairy graphs of genus zero is*

$$\begin{aligned} & R_{m_1 \dots m_r, d}^{0\pi}(x_1, \dots, x_r, z, u) \\ &= z\alpha_1\left(\frac{1}{z}\right) + \frac{1 - z^{d-2}u\alpha_1\left(\frac{1}{z}\right)}{z^{d-3}u} \sum_{l=1}^{+\infty} \frac{\mu(l) \ln \left(1 - (-1)^{(l-1)d} (z^{d-2}u)^l \alpha_l\left(\frac{1}{z}\right)\right)}{l}, \end{aligned}$$

where $\alpha_l\left(\frac{1}{z}\right) = \sum_{i=1}^r (-1)^{m_i(l-1)} x_i^l \left(\frac{1}{z}\right)^{m_i l}$.

Proof. This formula is obtained from (2.15) by change of variables: first using (2.16) and then (3.6). \square

Theorem 2.5. *The generating functions of the Euler characteristics of the Hodge splitting of the complex $\mathcal{E}_{0\pi}^{m_1, \dots, m_r, d}$ of hairy graphs of genus zero is*

$$\begin{aligned} & F_{m_1, \dots, m_r, d}^{0\pi}(x_1, \dots, x_r, u) \\ &= - \sum_{i=1}^r (-1)^{m_i} x_i + \left(\sum_{i=1}^r (-1)^{m_i} x_i - \frac{(-1)^d}{u} \right) \sum_{l=1}^{+\infty} \frac{\mu(l) \ln \left(1 - (-1)^d u^l \sum_{i=1}^r (-1)^{m_i} x_i^l\right)}{l}. \end{aligned}$$

Proof. Take $z = -1$ in the previous theorem. \square

2.2.3 Genus one

In the previous section we have computed the generating function for graph-complexes of genus zero. Here we will make computations in the genus one case. Figure 4 and Figure 5 are examples of graphs of genus one.

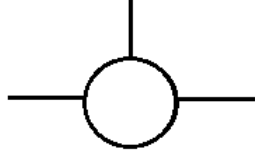


Figure 4: A graph of genus one

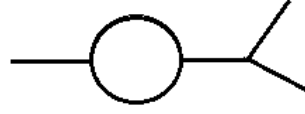


Figure 5: Another graph of genus one

The first goal will be to understand the Σ_n action on $H_*M_1(P_d^n)$. Notice that in this homology we can consider only hedgehogs: graphs whose external vertices are directly connected to the loop. This is because the other graphs are killed by the differential in homology (see [5]). A typical graph (or generator) in $H_*M_1(P_d^n)$ is the one of Figure 6.

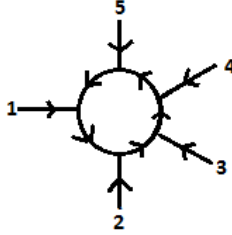


Figure 6: A graph in $H_*M_1(P_d^5)$

A hedgehog with n external vertices has $2n$ edges and n internal vertices, thus the homology $H_*M_1(P_d^n)$ is concentrated in the only degree $2n(d-1) - nd = n(d-2)$. Let D_n denote the dihedral group of symmetries of the unit circle with n points marked $e^{\frac{2k\pi i}{n}}$, $k = 0 \dots n-1$. One has an obvious homomorphism $D_n \rightarrow \Sigma_n$ corresponding to the permutation of the marked points. This map is an inclusion for $n \geq 3$. The hedgehog G whose external vertices are marked in the cyclic order $1, \dots, n$ is sent to itself times certain sign when acted on by elements of D_n . Denote by λ_n the character of D_n corresponding to this sign. (This character depends on d as well.) As a Σ_n module in graded vector spaces $H_*M_1(P_d^n)$ is the $n(d-2)$ -suspended induced representation

$$H_*M_1(P_d^n) \cong_{\Sigma_n} \Sigma^{n(d-2)} \text{Ind}_{D_n}^{\Sigma_n} \lambda_n. \quad (2.17)$$

Let us describe the character λ_n . Since D_n is generated by two elements: a $\frac{2\pi}{n}$ rotation and a reflection, it is enough to compute λ_n on those elements. For an element $\sigma \in D_n$, we will explicitly determine the sign $\lambda_n(\sigma)$ (using the Koszul sign of permutation taking into account the degrees of the elements in the orientation set) that appears in the equality $\sigma \cdot G = \lambda_n(\sigma)G$. To recall the orientation set is the union of edges, of degree $d-1$, and internal vertices, of degree $-d$. There are three possibilities:

- If σ is a rotation, then one can easily see that $\sigma \cdot G = (-1)^{d(n-1)}G$;
- If n is odd and σ is a reflection whose one vertex is fixed (see Figure 7), then $\sigma \cdot G = (-1)^{\frac{n+1}{2}d}G$;

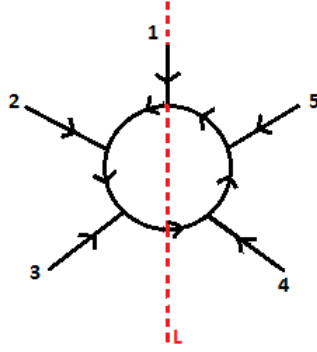


Figure 7: A reflection with respect to L ($n=5$)

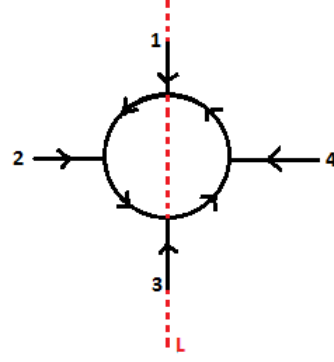


Figure 8: A reflection with respect to L ($n=4$)

- If n is even and σ is a reflection whose two vertices are fixed (see Figure 8), then one has $\sigma \cdot G = (-1)^{\frac{nd}{2}-1}G$.

In order to simplify computations, we will write λ_n in another form. Let $\text{sign}: D_n \rightarrow \{-1, 1\}$ be the signature representation restricted from Σ_n , and let $\text{or}: D_n \rightarrow \{-1, 1\}$ be the orientation representation. Notice that the latter representation concerns only reflections in D_n , that is, “or” is equal to -1 on reflections, and to 1 on rotations. From our computations we obtain

$$\lambda_n = \text{sign}^{\otimes d} \otimes \text{or}^{\otimes (n+d+1)}. \quad (2.18)$$

We will need the following well known fact [8]. To recall the cycle index sum is defined in Definition 1.3.

Lemma 2.6. *Let $f: H \rightarrow \Sigma_n$ be a group homomorphism and let $\rho^V: H \rightarrow GL(V)$ be a representation of H , then the cycle index sum of the induced representation can be expressed as*

$$Z_{\text{Ind}_H^{\Sigma_n} V}(p_1, p_2, \dots) = \frac{1}{|H|} \sum_{h \in H} \text{tr}(\rho^V(h)) \prod_l p_l^{j_l(f(h))}. \quad (2.19)$$

Let $RO_n \subseteq D_n$ denote the subset of D_n formed by rotations, and let RE_n denote the subset formed by reflections. Using the above lemma we get

$$Z_{\text{Ind}_{D_n}^{\Sigma_n} \lambda_n}(p_1, p_2, \dots) = \frac{1}{|D_n|} \sum_{\sigma \in D_n} \lambda_n(\sigma) \prod_l p_l^{j_l(\sigma)} \quad (2.20)$$

$$= \frac{1}{2n} \sum_{\sigma \in RO_n} \lambda_n(\sigma) \prod_l p_l^{j_l(\sigma)} + \frac{1}{2n} \sum_{\sigma \in RE_n} \lambda_n(\sigma) \prod_l p_l^{j_l(\sigma)}. \quad (2.21)$$

We will denote by $\overline{A}_n(p_1, p_2, \dots)$ and $\overline{B}_n(p_1, p_2, \dots)$ respectively the first and the second summands in (2.21).

Let $\mathbf{1}$ denote the trivial character of D_n . We compute first the cycle index sum $Z_{\text{Ind}_{D_n}^{\Sigma_n} \mathbf{1}}$. Then we will add some signs to describe $Z_{\text{Ind}_{D_n}^{\Sigma_n} \lambda_n}$.

$$Z_{\text{Ind}_{D_n}^{\Sigma_n} \mathbf{1}}(p_1, p_2, \dots) = A_n(p_1, p_2, \dots) + B_n(p_1, p_2, \dots),$$

where

$$A_n = \frac{1}{2n} \sum_{l|n} \varphi(l) p_l^{\frac{n}{l}} \quad \text{and} \quad B_n = \begin{cases} \frac{1}{2} p_1 p_2^{\frac{n-1}{2}} & \text{if } n \text{ odd,} \\ \frac{1}{4} (p_1^2 p_2^{\frac{n-2}{2}} + p_2^{\frac{n}{2}}) & \text{if } n \text{ even,} \end{cases}$$

where $\varphi(l)$ is the Euler's totient function that produces the number of positive integers less than or equal to l that are relatively prime to l . Explicitly $\varphi(l) = \sum_{a|l} \mu(a) \frac{l}{a}$.

Summing over $n \geq 1$, we get

$$Z_{\text{Ind}_D^{\Sigma_D \bullet} \mathbf{1}}(p_1, p_2, \dots) = -\frac{1}{2} \sum_{l \geq 1} \frac{\varphi(l) \ln(1 - p_l)}{l} + \frac{p_1^2 + p_2 + 2p_1}{4(1 - p_2)}. \quad (2.22)$$

On the other hand, from (2.18) we get

$$\overline{A}_n(p_1, p_2, \dots) = A_n(p_l \leftarrow (-1)^{d(l-1)} p_l),$$

$$\overline{B}_n(p_1, p_2, \dots) = (-1)^{n+d+1} B_n(p_l \leftarrow (-1)^{d(l-1)} p_l) = (-1)^{d+1} B_n(p_l \leftarrow (-1)^{d(l-1)+l} p_l).$$

Since $\sum_{n \geq 1} A_n$ and $\sum_{n \geq 1} B_n$ are respectively the first and second summands in (2.22), we get

$$Z_{\text{Ind}_D^{\Sigma_D \bullet} \lambda_\bullet}(p_1, p_2, \dots) = -\frac{1}{2} \sum_{l \geq 1} \frac{\varphi(l) \ln(1 - (-1)^{d(l-1)} p_l)}{l} + (-1)^{d+1} \frac{p_1^2 + (-1)^d p_2 - 2p_1}{4(1 - (-1)^d p_2)}. \quad (2.23)$$

From (2.17) and the fact that any genus 1 graph with k external vertices has complexity k ,

$$Z_{H_* M_1(P_d^\bullet)}(z, u; p_1, p_2, \dots) = Z_{\text{Ind}_D^{\Sigma_D \bullet} \lambda_\bullet}(p_l \leftarrow z^{(d-2)l} u^l p_l, l \in \mathbb{N}). \quad (2.24)$$

Recall the *Hodge splitting* from (2.7).

Theorem 2.7. *The generating function of the dimensions and of the Hodge splitting of the complex $\mathcal{E}_{1\pi}^{m_1, \dots, m_r, d}$ of hairy graphs of genus one is*

$$R_{m_1 \dots m_r d}^{1\pi}(x_1, \dots, x_r, z, u) = -\frac{1}{2} \sum_{l \geq 1} \frac{\varphi(l) \ln(1 - (-1)^{d(l-1)} z^{l(d-2)} u^l \alpha_l(\frac{1}{z}))}{l} + (-1)^{d+1} \frac{z^{2d-4} u^2 \alpha_1(\frac{1}{z})^2 + (-1)^d z^{2d-4} u^2 \alpha_2(\frac{1}{z}) - 2z^{d-2} u \alpha_1(\frac{1}{z})}{4(1 - (-1)^d z^{2d-4} u^2 \alpha_2(\frac{1}{z}))},$$

where $\alpha_l(\frac{1}{z}) = \sum_{i=1}^r (-1)^{m_i(l-1)} x_i^l (\frac{1}{z})^{m_i l}$.

Proof. This formula is obtained from (2.23) by change of variables: first using (2.24) and then (3.6). \square

Theorem 2.8. *The generating functions of the Euler characteristics of the Hodge splitting of the complex $\mathcal{E}_{1\pi}^{m_1, \dots, m_r, d}$ of hairy graphs of genus one is*

$$F_{m_1 \dots m_r d}^{1\pi}(x_1, \dots, x_r, u) = -\frac{1}{2} \sum_{l \geq 1} \frac{\varphi(l) \ln(1 - (-1)^d u^l \sum_{i=1}^r (-1)^{m_i} x_i^l)}{l} + (-1)^{d+1} \frac{u^2 (\sum_{i=1}^r (-1)^{m_i} x_i)^2 + (-1)^d u^2 \sum_{i=1}^r (-1)^{m_i} x_i^2 - 2(-1)^d u \sum_{i=1}^r (-1)^{m_i} x_i}{4(1 - (-1)^d u^2 \sum_{i=1}^r (-1)^{m_i} x_i^2)}.$$

Proof. Take $z = -1$ in the previous theorem. \square

3 Supercharacter of the symmetric group action on $\mathbf{Mod}(L_\infty)$

In Section 2, more precisely in Subsection 2.1, we computed the Euler characteristics for the Hodge splitting in the rational homotopy of string links. As a consequence of these computations, we obtain a curious result in the theory of operads: Theorem 0.4. The goal of this section is to prove that result.

The title of this section includes two terms: “supercharacter” and “ $\mathbf{Mod}(L_\infty)$ ” that we define now. Let us start with the first one. Let $M = (\oplus_i M_i, \partial)$ be a finite dimensional chain complex of Σ_k -modules over a ground field \mathbb{K} of characteristic 0. By the *supercharacter* we understand the character of the Σ_k action on the virtual representation $\mathcal{X}(M)$ defined as

$$\mathcal{X}(M) := \sum_i (-1)^i M_i.$$

The latter virtual representation is similar to the Euler characteristic in the sense that

$$\mathcal{X}(M) \simeq \mathcal{X}(H_*(M)),$$

that’s why we use this notation. The cycle index sum encoding the supercharacter of the Σ_k action on M can be defined as

$$Z_{\mathcal{X}(M)} = \sum_i (-1)^i Z_{M_i},$$

or equivalently as $Z_M|_{z=-1}$.

For a symmetric sequence of chain complexes $M = \{M(k)\}_{k \geq 0}$, we similarly define

$$Z_{\mathcal{X}(M)} := \sum_{k \geq 0} Z_{\mathcal{X}(M(k))}.$$

Define now $\mathbf{Mod}(L_\infty)$. First of all, L_∞ is the usual Koszul resolution for the operad \mathbf{Lie} . Similarly to \mathbf{Lie} , it is also a *cyclic operad* (see [9, 13] for the notion of “cyclic operad”). It is generated by operations of arity $n \geq 2$, and of degree $n - 2$. In arity 2 the generator is the usual brackets $[-, -]$, in arity three we have the bracket $[-, -, -]$, and so on. From any cyclic operad, one can construct the so called a *modular operad* (see [10, 13, 19]) using the well known adjunction

$$\mathbf{Mod}: \text{cyclic operads} \rightleftarrows \text{modular operads} : U$$

between cyclic operads and modular operads, the functor \mathbf{Mod} being the left adjoint to the forgetful functor U . Note that by definition every modular operad is in particular a cyclic operad.

Definition 3.1. *Given a cyclic operad O , its modular envelop is the modular operad $\mathbf{Mod}(O)$.*

Since L_∞ is a cyclic operad, one can then consider the modular envelop $\mathbf{Mod}(L_\infty)$. Theorem 0.4 announced in the introduction computes the cycle index sum $Z_{\mathcal{X}(\mathbf{Mod}(L_\infty))}$, which encodes the supercharacter of the symmetric group action on the components $\mathbf{Mod}(L_\infty)(k)$, $k \geq 1$. The case $k = 0$ is what is called the commutative operad graph-complex and was studied in [25].⁵

For the rest of this section, V will denote the r -dimensional vector space whose basis is the set of colours v_1, \dots, v_r , with each v_i being of degree m_i and of Hodge multi-degree $(\underbrace{0 \dots 0}_{i-1}, 1, \underbrace{0 \dots 0}_{r-i})$. Consider the symmetric sequence $V^{\otimes \bullet} = \{V^{\otimes n}\}_{n \geq 0}$. We will need to know the cycle index sum $Z_{V^{\otimes \bullet}}$. Let us start by introducing, for each $1 \leq i \leq r$, the one dimensional vector space V_i spanned by v_i , and consider the

⁵More precisely $\mathbf{Mod}(L_\infty)(0)$ is isomorphic to \mathbf{GC}_3 studied in [25], compare with Lemma 3.3 below.

symmetric sequence $V_i^{\otimes \bullet} = \{V_i^{\otimes n}\}_{n \geq 0}$. One can rewrite the vector space V in the form $V = V_1 \oplus \cdots \oplus V_r$. Therefore,

$$V^{\otimes n} = \bigoplus_{k_1 + \cdots + k_r = n} \text{Ind}_{\Sigma_{\bar{k}}}^{\Sigma_k} V_1^{\otimes k_1} \otimes \cdots \otimes V_r^{\otimes k_r}, \text{ where } \Sigma_{\bar{k}} = \Sigma_{k_1} \times \cdots \times \Sigma_{k_r}.$$

Recalling Definition 1.1 from Subsection 1.1 we have $V^{\otimes \bullet} = V_1^{\otimes \bullet} \widehat{\otimes} \cdots \widehat{\otimes} V_r^{\otimes \bullet}$, and by Lemma 1.5 one has

$$Z_{V^{\otimes \bullet}} = \prod_{i=1}^r Z_{V_i^{\otimes \bullet}}. \quad (3.1)$$

For $1 \leq i \leq r$ we will compute $Z_{V_i^{\otimes \bullet}}$. By noticing that the action of Σ_n on $V_i^{\otimes n}$ depends on the parity of m_i (for odd m_i the action is the sign representation, which means that if $\sigma \in \Sigma_n$, $x \in V_i^{\otimes n}$, then $\sigma x = \pm x$, and for even m_i it is the identity), by also noticing that V_i is a one dimensional vector space, it follows that $V_i^{\otimes \bullet}$ is the commutative unital operad "up to sign". One has

$$Z_{\text{Com}}(p_1, p_2, \dots) = \exp\left(\sum_{l \geq 1} \frac{p_l}{l}\right),$$

see for example [7, Section 5]. We deduce that

$$Z_{V_i^{\otimes \bullet}}(z, x_i; p_1, p_2, \dots) = Z_{\text{Com}}(p_l \leftarrow (-1)^{m_i(l-1)} x_i^l z^{m_i l} p_l) = \exp\left(\sum_{l \geq 1} (-1)^{m_i(l-1)} x_i^l z^{m_i l} \frac{p_l}{l}\right), \quad (3.2)$$

where the variable z is responsible for the usual homological degree and x_i is responsible for the i -th Hodge grading. The sign $(-1)^{m_i(l-1)}$ appears because a cycle of length l is an odd representation if and only if l is even. The factors x_i^l and $z^{m_i l}$ encode the fact that $V_i^{\otimes l}$ is concentrated in the Hodge multi-degree $(\underbrace{0 \dots 0}_{i-1}, l, \underbrace{0 \dots 0}_{r-i})$ and homological degree $m_i l$.

Combining (3.1) and (3.2), we have

$$Z_{V^{\otimes \bullet}}(z, x_1, \dots, x_r; p_1, p_2, \dots) = \exp\left(\sum_{l \geq 1} \alpha_l(z, x_1, \dots, x_r) \frac{p_l}{l}\right), \quad (3.3)$$

where

$$\alpha_l(z, x_1, \dots, x_r) = \sum_{i=1}^r (-1)^{m_i(l-1)} x_i^l z^{m_i l}. \quad (3.4)$$

For computations of the Euler characteristics we will need

$$\alpha_l(-1) = \sum_{i=1}^r (-1)^{m_i} x_i^l. \quad (3.5)$$

For any symmetric sequence $M(\bullet)$, we get

$$\begin{aligned} \dim \text{hom}_{\Sigma}(V^{\otimes \bullet}, M(\bullet)) &= \left\{ Z_{V^{\otimes \bullet}}\left(\frac{1}{z}; p_l \leftarrow l \frac{\partial}{\partial p_l}, l \in \mathbb{N}\right) Z_{M(\bullet)}(z; p_1, p_2, \dots) \right\} \Big|_{p_l=0} \\ &= Z_{M(\bullet)}\left(z; p_l \leftarrow \alpha_l\left(\frac{1}{z}, x_1, \dots, x_r\right), l \in \mathbb{N}\right). \end{aligned} \quad (3.6)$$

Notation "dim" stays for the generating function of dimensions that takes into account both homological degree (with z responsible for it) and the Hodge degrees (x_1, \dots, x_r are the responsible variables).

The hairy graph-complex $\mathcal{E}_{\pi}^{m_1, \dots, m_r, d}$ we recalled at the beginning of Subsection 2.2 has exactly this form:

$$\mathcal{E}_{\pi}^{m_1, \dots, m_r, d} \cong \text{hom}_{\Sigma}(V^{\otimes \bullet}, M(P_d^{\bullet})), \quad (3.7)$$

where $M(P_d^k)$ is the graph-complex from Subsection 2.2.1.

Theorem 3.2. *The supercharacter of the symmetric group action on the graph-complexes $\{M(P_d^k)\}_{k \geq 1}$ is described by the cycle index sum*

$$Z_{\mathcal{X}(M(P_d^\bullet))}(u; p_1, p_2, p_3, \dots) = \sum_{k, l, j \geq 1} \frac{\mu(k)}{kj} S_j \left(-\frac{1}{l} \sum_{a|l} \mu\left(\frac{l}{a}\right) p_{ak} \right) \left(\frac{(-1)^{d-1} l u^{kl}}{F_l(u^k)} \right)^j +$$

$$\sum_{k, l \geq 1} \frac{\mu(k)}{kl} \left(\sum_{a|l} \mu\left(\frac{l}{a}\right) p_{ak} \right) \ln(F_l(u^k)),$$

where the variable u is as usual responsible for complexity.

Proof. Applying (3.6) to $M(\bullet) = M(P_d^\bullet)$ and taking $z = -1$, we get

$$F_{m_1 \dots m_r d}^\pi(x_1, \dots, x_r, u) = Z_{\mathcal{X}(M(P_d^\bullet))}(u, p_l \leftarrow \alpha_l(-1)). \quad (3.8)$$

Thus to reconstruct $Z_{\mathcal{X}(M(P_d^\bullet))}$ from $F_{\pi^{m_1 \dots m_r d}}$ we need to replace each occurrence of $\alpha_l(-1) = \sum_{i=1}^r (-1)^{m_i} x_i^l$ back to p_l . Using the result of Theorem 0.3 and the fact that $E_l(x) = \frac{1}{l} \sum_{a|l} \mu\left(\frac{a}{l}\right) x^a$ we get the result. \square

Consider now the modular envelop $\mathbf{Mod}(L_\infty)$. It is easy to notice that graph-complexes $M(P_d^\bullet)$ are closely related to the components of $\mathbf{Mod}(L_\infty)$. More precisely if we choose $d = 3$ and also tensor each component $M(P_d^k)$ with the sign representation and take a shift in degree (desuspension) by k we get $\mathbf{Mod}(L_\infty)(k)$.

Lemma 3.3. *For any $k \geq 1$,*

$$\mathbf{Mod}(L_\infty)(k) \cong_{\Sigma_k} \Sigma^{-k} M(P_3^k) \otimes \text{sign}. \quad (3.9)$$

Proof. Combinatorially $\mathbf{Mod}(L_\infty)(k)$ is a graph-complex consisting of exactly the same graphs as $M(P_3^k)$, so we only need to work out the signs and degrees properly. The operad L_∞ is cyclic and is freely generated by operations of arity l , $l \geq 2$, which have degree $l - 2$ and the sign action of Σ_{l+1} . Graphically such operations correspond to vertices of arity $l + 1$. As a conclusion, to orient a graph $G \in \mathbf{Mod}(L_\infty)(k)$ we need to order its vertices, where a vertex v of valence $|v|$ is considered as element of degree $|v| - 3$, and for each vertex to order edges adjacent to it. Changing the order of adjacent edges at any vertex gives the sign of permutation; changing the order of vertices gives the Koszul sign of permutation. Now when we look at a graph $G \in M(P_3^k)$, it is oriented by ordering the set of its vertices (considered as elements of degree -3) and edges (considered as elements of degree 2, therefore their placement in the orientation set can be ignored), and by orienting all edges. Changing orientation of an edge gives a negative sign. Now, we replace each edge in the orientation set by its two half-edges in the order – first source, second target. Then we change the order of the elements in the orientation set so that the vertices and adjacent to it half-edges come in one block – first the vertex than half-edges. The combined block corresponding to any vertex v has degree exactly $|v| - 3$. Notice however that k half-edges, corresponding to the external vertices, don't appear in any such block. These half-edges get annihilated with $\Sigma^{-k} \text{sign}$ in (3.9). \square

Now we have all ingredients to prove Theorem 0.4: Lemma 3.3 and Theorem 3.2. In the cycle index sum of the latter result, there is a variable u , which is responsible for complexity. Since $\mathbf{Mod}(L_\infty)$ is a modular operad, it is more natural to consider the splitting by genus rather than by complexity. We use the variable w as the one responsible for the genus.

Proof of Theorem 0.4. It follows from (3.9) that

$$Z_{\mathcal{X}(\mathbf{Mod}(L_\infty)(\bullet))}(u; p_1, p_2, p_3, \dots) = Z_{\mathcal{X}(M(P_3^\bullet))}(u; -p_1, -p_2, -p_3, \dots). \quad (3.10)$$

Also because we are interested in the splitting by genus rather than by complexity, it follows from (2.9) that we need to make the change of variables $u \leftarrow w$, $p_l \leftarrow \frac{p_l}{w^l}$, and in addition to it multiply the result by w . (Compare with (2.10).) Combining it with (3.10) we get

$$Z_{\mathcal{X}(\mathbf{Mod}(L_\infty)(\bullet))}(w; p_1, p_2, p_3, \dots) = w Z_{\mathcal{X}(M(P_3^\bullet))}(u \leftarrow w; p_l \leftarrow -\frac{p_l}{w^l}, l \in \mathbb{N}).$$

To finish the proof we apply Theorem 3.2. □

Remark 3.4. *In case of even d , the sequence $M(P_d^\bullet)$ can also be interpreted as a certain twisted modular envelop of a suspended L_∞ operad. The suspension functor preserves usual operads, but it turns cyclic operads into anticyclic ones [9, Section 2]. For such operads one can define a twisted modular envelop which is a twisted modular operad [19, Section 5].*

A Appendix. Tables of Euler characteristics

Here we present results of computer calculations which were produced using Mathematica. Recall the splitting of the complex $\mathcal{E}_\pi^{m_1, \dots, m_2, d}$ from (2.6). One can split it again into a direct sum

$$\mathcal{E}_\pi^{m_1, \dots, m_2, d} = \bigoplus_{g \geq 0} \bigoplus_{s_1, \dots, s_r, t} \mathcal{E}_{g\pi s_1, \dots, s_r, t}^{m_1, \dots, m_2, d}.$$

Let $\chi_{s_1 \dots s_r t}^{g\pi}$ denote the Euler characteristic of each summand in that splitting. The following tables furnish results of $\chi_{s_1 s_2 t}^{g\pi}$ (that is, in the case $r = 2$) for genus $g \in \{0, 1, 2, 3\}$ with m_1, m_2 and d odd. Recall the formula $g + s_1 + s_2 = t + 1$ from (2.9).

t	Hodge degree s_2																							
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 1: Table of Euler characteristics $\chi_{s_1 s_2 t}^{g\pi}$ by genus $g = 0$, complexity t and Hodge degree s_2 of $\pi_* \text{Emb}_c(\coprod_{i=1}^2 \mathbb{R}^{m_i}, \mathbb{R}^d) \otimes \mathbb{Q}$ for m_1, m_2 and d odd ($s_1 = t - s_2 + 1$).

t	Hodge degree s_2																							
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
7	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0
8	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0
9	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0
10	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0
11	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0
12	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
13	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
14	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
15	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
16	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
17	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
18	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
19	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
20	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
21	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
22	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
23	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Table 2: Table of Euler characteristics $\chi_{s_1 s_2 t}^{g\pi}$ by genus $g = 1$, complexity t and Hodge degree s_2 of $\pi_* \overline{\text{Emb}}_c(\coprod_{i=1}^2 \mathbb{R}^{m_i}, \mathbb{R}^d) \otimes \mathbb{Q}$ for m_1, m_2 and d odd ($s_1 = t - s_2$).

t	Hodge degree s_2																						
1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	-1	-1	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	-1	-1	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	1	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	-1	-1	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	2	2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	-2	-2	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	2	2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	-2	-2	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	2	2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	-2	-2	-2	2	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	3	3	3	-3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	-3	-3	-3	3	-3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	3	3	3	-3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	-3	-3	-3	3	-3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	3	3	3	-3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	-3	-3	-3	3	-3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	4	4	4	-4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	-4	-4	-4	4	-4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	4	4	4	-4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	-4	-4	-4	4	-4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	4	4	4	-4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3: Table of Euler characteristics $\chi_{s_1 s_2 t}^{g\pi}$ by genus $g = 2$, complexity t and Hodge degree s_2 of $\pi_* \text{Emb}_c(\coprod_{i=1}^2 \mathbb{R}^{m_i}, \mathbb{R}^d) \otimes \mathbb{Q}$ for m_1, m_2 and d odd ($s_1 = t - s_2 - 1$).

t	Hodge degree s_2																							
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 4: Table of Euler characteristics $\chi_{s_1 s_2 t}^{g\pi}$ by genus $g = 3$, complexity t and Hodge degree s_2 of $\pi_* \overline{\text{Emb}}_c(\coprod_{i=1}^2 \mathbb{R}^{m_i}, \mathbb{R}^d) \otimes \mathbb{Q}$ for m_1, m_2 and d odd ($s_1 = t - s_2 - 2$).

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Institut de Recherche en Mathématique et Physique (IRMP)
 Université Catholique de Louvain, Chemin du Cyclotron 2
 1348 Louvain-la-Neuve, Belgique
E-mail address: arnaud.songhafouo@uclouvain.be

Department of Mathematics, Kansas State University
 138 Cardwell Hall, Manhattan, KS 66506, USA
E-mail address: turchin@ksu.edu